

# Gernot Hoffmann

## Perspective Rectification for Images



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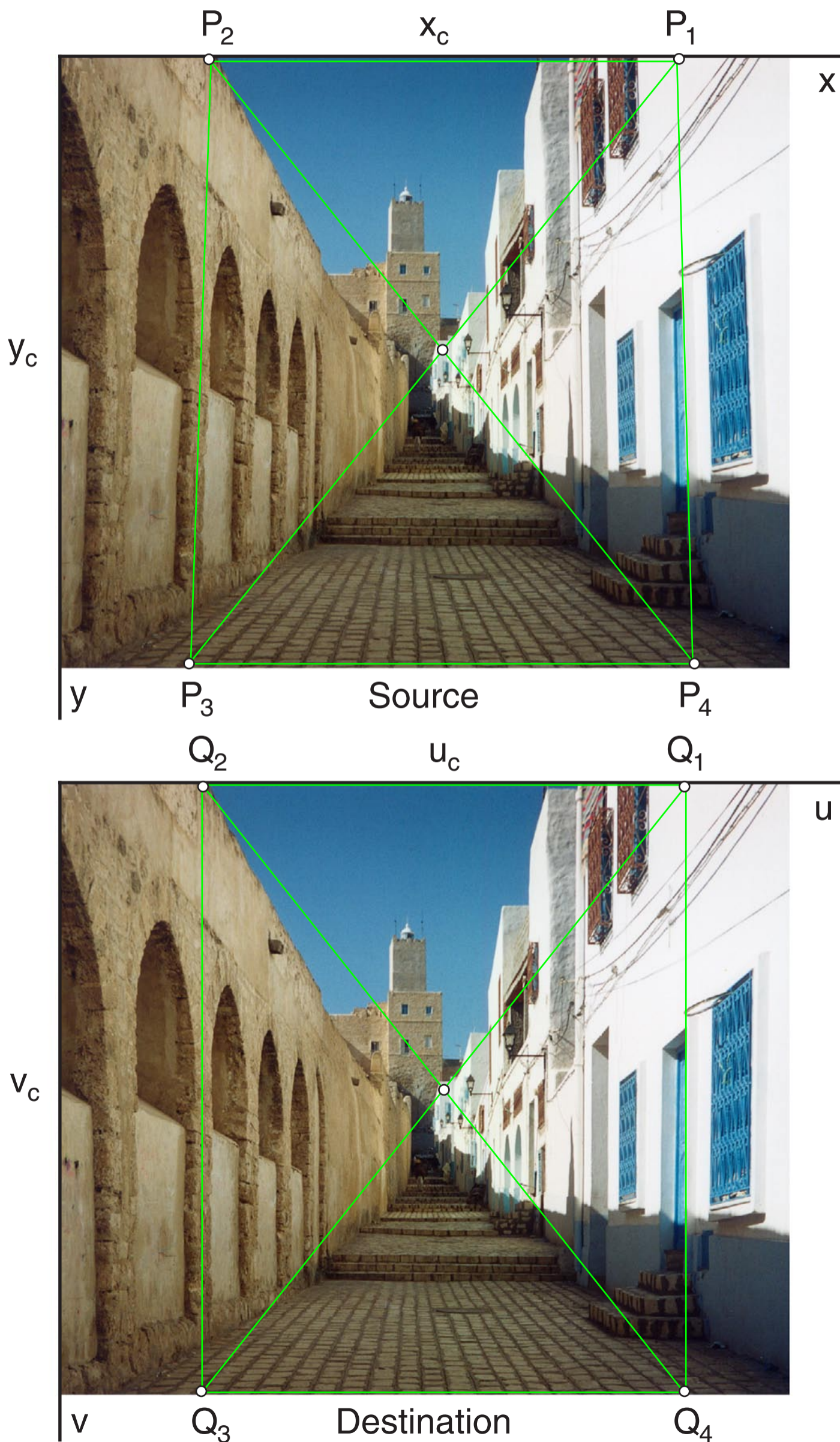
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Best view 72 dpi / zoom 100% or 200%

# 1. Introduction

A general quadrilateral is mapped to a rectangle (or to another quadrilateral). This defines the transformation for the whole image.

The destination image is filled by regular scanlines, taking the values from the source image by bicubic interpolation.



## 2.1 Mathematics / Mapping

General perspective mapping is well-known. The formulas are valid for mapping from 3D to 2D (11 parameters) or like here for mapping from 2D to 2D (8 parameters).

$$\begin{aligned}\mathbf{x} &= (x, y)^T \\ \mathbf{u} &= (u, v)^T \\ \mathbf{a} &= (a_u, a_v)^T \\ \mathbf{b} &= (b_u, b_v)^T \\ \mathbf{c} &= (c_u, c_v)^T\end{aligned}$$

$$x = \frac{a_0 + \mathbf{a}^T \mathbf{u}}{1 + \mathbf{c}^T \mathbf{u}}$$

$$y = \frac{b_0 + \mathbf{b}^T \mathbf{u}}{1 + \mathbf{c}^T \mathbf{u}}$$

$$a_0 + \mathbf{a}^T \mathbf{u} + 0 + 0 + 0 - x \mathbf{c}^T \mathbf{u} = x$$

$$0 + 0 + 0 + b_0 + \mathbf{b}^T \mathbf{u} - y \mathbf{c}^T \mathbf{u} = y$$

Mapping four points delivers eight linear equations

$$\mathbf{p} = (a_0, a_u, a_v, b_0, b_u, b_v, c_u, c_v)^T$$

$$\mathbf{r} = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)^T$$

$$\mathbf{A}\mathbf{p} = \mathbf{r}$$

$$\mathbf{A} = \begin{bmatrix} 1 & u_1 & v_1 & 0 & 0 & 0 & -x_1 u_1 & -x_1 v_1 \\ 1 & u_2 & v_2 & 0 & 0 & 0 & -x_2 u_2 & -x_2 v_2 \\ 1 & u_3 & v_3 & 0 & 0 & 0 & -x_3 u_3 & -x_3 v_3 \\ 1 & u_4 & v_4 & 0 & 0 & 0 & -x_4 u_4 & -x_4 v_4 \\ 0 & 0 & 0 & 1 & u_1 & v_1 & -y_1 u_1 & -y_1 v_1 \\ 0 & 0 & 0 & 1 & u_2 & v_2 & -y_2 u_2 & -y_2 v_2 \\ 0 & 0 & 0 & 1 & u_3 & v_3 & -y_3 u_3 & -y_3 v_3 \\ 0 & 0 & 0 & 1 & u_4 & v_4 & -y_4 u_4 & -y_4 v_4 \end{bmatrix}$$

Simplified center and estimation of the half edge lengths  $e_u$ ,  
The true center is the intersection of the diagonals (next pag

$$u_c = x_c = (x_1 + x_2 + x_3 + x_4) / 4$$

$$v_c = y_c = (y_1 + y_2 + y_3 + y_4) / 4$$

$$e_u = [(x_1 - x_2) + (x_4 - x_3)] / 4$$

$$e_v = [(y_4 - y_1) + (y_3 - y_2)] / 4$$

$$u_1 = u_c + e_u \quad v_1 = v_c - e_v$$

$$u_2 = u_c - e_u \quad v_2 = v_c - e_v$$

$$u_3 = u_c - e_u \quad v_3 = v_c + e_v$$

$$u_4 = u_c + e_u \quad v_4 = v_c + e_v$$

## 2.2 Mathematics / Center

The true center is the intersection of the diagonals.

$$\mathbf{x}_{31} = \mathbf{x}_3 - \mathbf{x}_1$$

$$\mathbf{x}_{42} = \mathbf{x}_4 - \mathbf{x}_2$$

$$\mathbf{x}_{21} = \mathbf{x}_2 - \mathbf{x}_1$$

$$\mathbf{x} = \mathbf{x}_1 + s\mathbf{x}_{31}$$

$$\mathbf{x} = \mathbf{x}_2 + t\mathbf{x}_{42}$$

$$s\mathbf{x}_{31} - t\mathbf{x}_{42} = \mathbf{x}_{21}$$

$$sx_{31} - tx_{42} = x_{21}$$

$$sy_{31} - ty_{42} = y_{21}$$

Solve by *Cramer's rule*

$$D = -x_{31}y_{42} + x_{42}y_{31}$$

$$S = -x_{21}y_{42} + x_{42}y_{21}$$

$$s = S/D$$

$$\mathbf{x}_c = \mathbf{x}_1 + s\mathbf{x}_{31}$$

## 2.3 Mathematics / Scaling

If coordinates  $x, y, u, v$  are given in millimeters or pixels then it is most likely that the linear equation system is ill-conditioned. Especially products like  $x_i u_k$  create huge numbers.

As a result, the determinant of the system might become quite large, e.g.  $10^{22}$  for the next example.

Mainly for PostScript, which works in some interpreters by fixed point arithmetic, proper scaling can help to improve the accuracy.

Use  $\mathbf{x}$  and  $\mathbf{u}$  in the units [mm] or [pixel] but use scaled values for the calculation of the parameters, e.g. by  $s=500$ :

$$x/s = \frac{a_0 + \mathbf{a}^T(\mathbf{u}/s)}{1 + \mathbf{c}^T(\mathbf{u}/s)}$$

$$y/s = \frac{b_0 + \mathbf{b}^T(\mathbf{u}/s)}{1 + \mathbf{c}^T(\mathbf{u}/s)}$$

Solve linear equation.

For the application use  $\mathbf{x}$  and  $\mathbf{u}$  in the units [mm] or [pixel] but modify the formula:

$$x = s \frac{a_0 + (\mathbf{a}^T/s)\mathbf{u}}{1 + \mathbf{c}^T\mathbf{u}/s}$$

$$y = s \frac{b_0 + (\mathbf{b}^T/s)\mathbf{u}}{1 + (\mathbf{c}^T/s)\mathbf{u}}$$



## 3.2 Perspective Transform by PostScript / Code

```
%!PS-Adobe-3.0    EPSF-3.0
%%BoundingBox:    0 0 596 842
%%Creator:        Gernot Hoffmann
%%Title:          Gen.PerspecScale
%%CreationDate:   August 15 2005/2003

% Define perspective 2D-2D transform by 4 corners
% of a rectangle in x,y and the quadrilateral in u,v

% General perspective mapping in homogeneous coordinates,

% Draw perspective text

% [u]    [ ax  ay  a0 ] [x]
% [v] = [ bx  by  b0 ]*[y]
% [w]    [ cx  cy  1  ] [1]

% Rearranged
% u = a0+ax*x+ay*y
% v = b0+bx*x+by*y
% w = 1+cx*x+cy*y

% u = u/w
% v = v/w

% /scal 500 def % is defined below

/mm {2.834646 mul } def

/N 8 def
/n1 N 1 sub def

/x1 0 mm def
/y1 0 mm def
/x2 210 mm def
/y2 0 mm def
/x3 210 mm def
/y3 297 mm def
/x4 0 mm def
/y4 297 mm def

/u1 20 mm def
/v1 20 mm def
/u2 190 mm def
/v2 30 mm def
/u3 200 mm def
/v3 177 mm def
/u4 10 mm def
/v4 277 mm def
```

## 3.3 Perspective Transform by PostScript / Code

```
/FillMat
{% retain x1 y1 ... x4 y4 for box; use r t
/r1 x1 scal div def
/r2 x2 scal div def
/r3 x3 scal div def
/r4 x4 scal div def
/s1 y1 scal div def
/s2 y2 scal div def
/s3 y3 scal div def
/s4 y4 scal div def
/u1 u1 scal div def
/u2 u2 scal div def
/u3 u3 scal div def
/u4 u4 scal div def
/v1 v1 scal div def
/v2 v2 scal div def
/v3 v3 scal div def
/v4 v4 scal div def
/ANN
[ 1 r1 s1 0 0 0 r1 u1 mul neg s1 u1 mul neg
  1 r2 s2 0 0 0 r2 u2 mul neg s2 u2 mul neg
  1 r3 s3 0 0 0 r3 u3 mul neg s3 u3 mul neg
  1 r4 s4 0 0 0 r4 u4 mul neg s4 u4 mul neg
  0 0 0 1 r1 s1 r1 v1 mul neg s1 v1 mul neg
  0 0 0 1 r2 s2 r2 v2 mul neg s2 v2 mul neg
  0 0 0 1 r3 s3 r3 v3 mul neg s3 v3 mul neg
  0 0 0 1 r4 s4 r4 v4 mul neg s4 v4 mul neg ] def
/YN [ u1 u2 u3 u4 v1 v2 v3 v4 ] def
/XN N array def
/TParams [ (a0) (ax) (ay) (b0) (bx) (by) (cx) (cy) ] def
} def

/AParams
{
/a0 XN 0 get def
/ax XN 1 get scal div def
/ay XN 2 get scal div def
/b0 XN 3 get def
/bx XN 4 get scal div def
/by XN 5 get scal div def
/cx XN 6 get scal div def
/cy XN 7 get scal div def
} def

% Path transforms by Roger Willcocks

/path 10000 array def
/pathindex 0 def

% Push value onto path array (marked as executable)

/Tpush
{ cvx
  path pathindex 3 -1 roll put
  /pathindex pathindex 1 add def
} def

% Pop x and y from op stack, transform by 'mat' above
% push result onto path array
```

## 3.4 Perspective Transform by PostScript / Code

```
/Xform
{ 10 dict
  begin
  /y exch def
  /x exch def
  mat 0 get x mul mat 1 get y mul add mat 2 get add
  mat 3 get x mul mat 4 get y mul add mat 5 get add
  mat 6 get x mul mat 7 get y mul add mat 8 get add
  /w exch def
  /v exch def
  /u exch def
  v w div scal mul
  u w div scal mul
  end
  Tpush Tpush
} def

% Transform current path by transform 'mat'
/Transformpath
{% matrix currentmatrix % save CTM; Can cause problems !
  % matrix defaultmatrix setmatrix
  /pathindex 0 def
  flattenpath
  { Xform /moveto Tpush }
  { Xform /lineto Tpush }
  { x }
  {/closepath Tpush } pathforall
  newpath
% execute the path array to create transformed path
  path 0 pathindex getinterval cvx exec
  % setmatrix % restore CTM
} def

/stroke
{ strokepath
  Transformpath
//fill
} def

/fill
{ Transformpath
//fill
} def

/show
{ true charpath
  fill
} def

/i 0 def /j 0 def /k 0 def
/dA 0 def /max 0 def /s 0 def
/h 0 def /q 0 def
/aik 0 def /akk 0 def
/bik 0 def /bkk 0 def
/pa 0 def /ca 0 def
```



## 3.5 Perspective Transform by PostScript / Code

```
/m
% Memory map; input i,k on stack; output N*i+k
{  exch
  N mul add
} bind def

/HoGaussP
{% Solve ANN*XN=YN for XN and det(ANN)
% N equations for N unknowns
% Overwrite all arrays
% No error detection
% Originally by H.R.Schwartz: Numerische Mathematik, B.G.Teubner, 1993
% Modified by G.Hoffmann for zero-based onedimensional arrays for PS
% Uses external memory mapping function m
/pa N array def
/ca N array def
/n1 N 1 sub def
/dA 1 def
0 1 n1 1 sub
{/k exch def
/max 0 def
pa k 0 put
k 1 n1
{/i exch def
/s 0 def
k 1 n1
{/j exch def
/s s ANN i j m get abs add def
} for %j
/bik ANN i k m get abs def
/q bik s div def
q max gt {/max q def pa k i put } if
} for %i
pa k get k ne
{/dA dA neg def
0 1 n1
{/j exch def
/h ANN k j m get def
ANN k j m ANN pa k get j m get put
ANN pa k get j m h put
} for %j
} if
/dA dA ANN k k m get mul def
k 1 add 1 n1
{/i exch def
/aik ANN i k m get def
/akk ANN k k m get def
/bkk akk abs def
bkk 1 lt
{/bik aik abs def
} if
ANN i k m aik akk div put
k 1 add 1 n1
{/j exch def
ANN i j m ANN i j m get ANN i k m get ANN k j m get mul sub put
} for %j
} for %i
} for %k
/dA dA ANN n1 n1 m get mul def
```

## 3.6 Perspective Transform by PostScript / Code

```
0 1 n1 1 sub
{/k exch def
  pa k get k ne
  {/h YN k get def
    YN k YN pa k get get put
    YN pa k get h put
  } if
} for %k
0 1 n1
{/i exch def
  ca i YN i get put
  0 1 i 1 sub
  {/j exch def
    ca i ca i get ANN i j m get ca j get mul sub put
  } for %j
} for %i
n1 -1 0
{/i exch def
  /s ca i get def
  i 1 add 1 n1
  {/k exch def
    /s s ANN i k m get YN k get mul sub def
  } for %k
  /bik ANN i i m get abs def
  YN i s ANN i i m get div put
} for %i
0 1 n1
{/i exch def
  XN i YN i get put
} for %i
} def % HoGaussP

/Box
{x1 y1 moveto
 x2 y2 lineto
 x3 y3 lineto
 x4 y4 lineto
 closepath
 stroke
} def

/Box2
{ 10 mm 10 mm moveto
 200 mm 10 mm lineto
 200 mm 287 mm lineto
 10 mm 287 mm lineto
 closepath
 stroke
} def
```

## 3.7 Perspective Transform by PostScript / Code

```
/Shownum
% Draw number by string
% Version May 15 2004 / uses rounding
%
% tx0 ty0 nu: on stack
%
% tx0      Position      not overwritten
% ty0
% nu      Input number   not overwritten   nu =+-999999
% fh      Font height    not overwritten
% tms     Mantissa number of characters    tms=0...6 Global
% tms=3   Example
% input  -23.56789  -999.99    0.4567  9999.123456
% result -23.568    -999.990   0.467   9999.123
% Postscript number to string is not well defined
% e.g. 1E-5 instead of 0.00001
% We use a straightforward BCD conversion.
% This is always affected by round-off errors
% because of 32-bit arithmetic
% Results are different, depending on the interpreter
{ /nu  exch def
  /ty0 exch def
  /tx0 exch def
  /tfw fh 0.6 mul def          % character distance
  /tna nu abs 10 tms neg exp 0.500001 mul add def
  /tdec 1E5 def
  /tchr 1 string def
  tna 999999.1 lt             % larger number replaced by #
  /tmm true def              % sign
  {/tx0 tx0 tfw 6 mul sub def
    /tz 0 def
    1 1 5                     % first 5 digits, no leading 0
    { pop
      /tk 0 def
      { tna tdec gt {/tna tna tdec sub def /tk tk 1 add def}{exit} ifelse
    } loop
      tk 0 ne {/tz tz 1 add def} if
      tz 0 ne
      { tx0 ty0 moveto tk tchr cvs show
      } if
      tz 1 eq nu 0 lt and tmm and          % minus   Correction 28/03/2005
      { tx0 tfw 0.7 mul sub ty0 moveto (-) show
        /tmm false def
      } if
      /tdec tdec 0.1 mul def
      /tx0 tx0 tfw add def
    } for
      /tk 0 def                          % leading 0
      { tna tdec gt {/tna tna tdec sub def /tk tk 1 add def}{exit} ifelse
    } loop
      tmm nu 0 lt and                    % minus
      { tx0 tfw 0.7 mul sub ty0 moveto (-) show
      } if
      tx0 ty0 moveto tk tchr cvs show
      /tdec tdec 0.1 mul def
      /tx0 tx0 tfw add def
    tms 0 gt                             % for float
    { tx0 ty0 moveto (.) show
      /tx0 tx0 tfw 0.5 mul add def
    } for
    1 1 tms
  { pop
```

## 3.8 Perspective Transform by PostScript / Code

```
/tk 0 def
{ tna tdec gt {/tna tna tdec sub def /tk tk 1 add def}{exit} ifelse
} loop
  tx0 ty0 moveto tk tchr cvs show
/tdec tdec 0.1 mul def
/tx0 tx0 tfw add def
} for
} if
}{ tx0 tfw sub ty0 moveto (#) show} ifelse
} def

% Unity transform
/scal 1 def
/mat
[ 1 0 0
  0 1 0
  0 0 1 ] def
0 setgray
0.8 mm setlinewidth
Box
0.4 mm setlinewidth
105 mm 105 mm 105 mm 0 360 arc stroke

% Perspective transform
/scal 500 def
FillMat
HoGaussP
AParams
/mat
[ ax ay a0
  bx by b0
  cx cy 1 ] def
0.0 0.5 0.8 setrgbcolor
Box
105 mm 105 mm 105 mm 0 360 arc stroke

/fh 10 mm def
/Helvetica findfont fh scalefont setfont
/xa 10 mm def
/ya 280 mm def
/buf 20 string def

/d1x fh 2.0 mul def
/d2x fh 4.5 mul def

/tms 6 def

0 1 n1
{/i exch def
  xa ya moveto TParams i get show
  xa d1x add ya moveto (=) show
  xa d2x add ya XN i get Shownum
/ya ya fh sub def
} for
xa ya moveto (det) show
xa d1x add ya moveto (=) show
xa d2x add ya dA Shownum

showpage
```

## 4. References

More References are in [1]

- [1] G.Hoffmann  
Planar Projections  
<http://docs-hoffmann.de/project18032004.pdf>
  
- [2] G.Hoffmann  
Rectification by Photogrammetry  
<http://docs-hoffmann.de/sans04012001.pdf>
  
- [3] G.Hoffmann  
Bicubic Interpolation (and other methods)  
<http://docs-hoffmann.de/bicubic03042002.pdf>

This doc:

Perspective Rectification for Images

<http://docs-hoffmann.de/persprect13052005.pdf>

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May 13 / 2005 — February 18 / 2013  
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