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Faddejev Algorithm
for
Eigenvalues and Eigenvectors

Contents

1.1 Introduction

Many tasks in physics, mechanics and computer graphics result in the so-called eigenvalue problem for matrices.
Given is a quadratic matrix with nxn elements, which are here real numbers. The matrix is for the general case not symmetric. This is the Eigenwertproblem: find solutions λ and $\mathbf{x}$ for equation (1).

(1) $\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x}$

(2) $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$

(3) $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

(4) $(\mathbf{A} - \lambda_k\mathbf{I})\,\mathbf{n}_k = \mathbf{0}$

Equation (2) is a homogeneous system of n linear equations for n unknowns. Non-trivial solutions can be expected if the determinant (3) is zero. This delivers primarily a polynomial function $f(\lambda)$. The eigenvalues λ are found as the roots of $f(\lambda)=0$, which is called the characteristic equation.

Equation (2) results for each eigenvalue λ_k in equation (4) for the eigenvector $\mathbf{x}=\mathbf{n}_k$. An eigenvector can be multiplied by any factor. A unit eigenvector has the Euclidian norm one. Complex eigenvectors are here normalized for one real value one.

The author is programming a library for linear algebra and related topics in PostScript, for simplicity called the ‘Matrix Library‘ [6], [7].
PostScript is generally based on single float precision - a rather limiting fact. Programming numerically algorithms is hampered by lacking structures as in Fortran, Pascal or C.
PostScript is used because of excellent graphics and typographical quality for vector graphics.
It is not expected that highly complex tasks of linear algebra have to be handled.

On this occasion, the author remembered the otherwise hardly known algorithm by *Leverrier*, improved by *Faddejev* [1]. This is here simply called Faddejev algorithm.
Just a few matrix operations deliver the determinant, the inverse matrix and the coefficients of the characteristic equation. Once the roots are found - here by *Newton-Bairstow* [2] - the eigenvectors are calculated by using intermediate matrices from the previous process.

Solving the eigenvalue problem via the characteristic equation is nowadays not considered as professional. One of the better alternatives is the QR-Algorithm [2]. Programming this in PostScript for non-symmetric matrices with possible complex solutions is for the author practically impossible.
On the other hand, for small matrices, without any symmetry, for real or complex distinct eigenvalues and eigenvectors, the Faddejev method is straightforward.

Practical problems arise for simple cases, e.g. the calculation of the eigenvalues for a 3x3 identity matrix. The three eigenvalues are all 1. A vector base of three linearly independent eigenvectors can be constructed, but it cannot be found by numerical calculations.

Faddejev‘s book contains a major bug, concerning the eigenvector extraction. The problem could be solved – slowly but safely, if the task is not affected by general problems like multiple eigenvalues, concerning the linearly independent vector base. This document shows a couple of examples.

2.1 Faddejev algorithm / General

Some simple matrix operations deliver the necessary informations for the calculation of the determinant, the inverse matrix and the coefficients of the characteristic equation. Once the roots are found (eigenvalues), it should be possible to extract the respective eigenvectors from a matrix **Q** , which is easily calculated, based on previous intermediate results. Unfortunately, the situation is not as simple as explained by Faddejev.

Faddejev algorithm for an $n \times n$ matrix **A**:

I is an $n \times n$ identity matrix. 'tr' means 'trace':

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$

$$\begin{array}{llll}
\mathbf{A}_1 = \mathbf{A} & p_1 = \mathrm{tr}(\mathbf{A}_1)/1 & \mathbf{B}_1 = \mathbf{A}_1 - p_1\,\mathbf{I} \\
\mathbf{A}_2 = \mathbf{A}\mathbf{B}_1 & p_2 = \mathrm{tr}(\mathbf{A}_2)/2 & \mathbf{B}_2 = \mathbf{A}_2 - p_2\,\mathbf{I} \\
\ldots & \ldots & \ldots \\
\mathbf{A}_n = \mathbf{A}\mathbf{B}_{n-1} & p_n = \mathrm{tr}(\mathbf{A}_n)/n & \mathbf{B}_n = \mathbf{A}_n - p_n\,\mathbf{I}
\end{array}$$

Always valid:

$\mathbf{B}_n = \mathbf{0}$

Determinant and inverse matrix:

$\det \mathbf{A} = (-1)^{n-1}\, p_n$

If $p_n \neq 0$:

$\mathbf{A}^{-1} = \mathbf{B}_{n-1} / p_n$

Characteristic equation:

$$f(\lambda) = (-1)^n\,[\lambda^n - p_1\,\lambda^{n-1} - p_2\,\lambda^{n-2} - \ldots - p_{n-1}\,\lambda^1 - p_n] = 0$$

The n eigenvalues λ_k are the roots of the characteristic equation.
They can be multiple. For instance $n = 5$ and triple λ_2:
$\lambda_1, \lambda_2, \lambda_3{=}\lambda_2, \lambda_4{=}\lambda_2, \lambda_5$

The matrix $\mathbf{Q}_k$ contains eigenvectors for λ_k in columns:

$$\mathbf{Q}_k = \lambda_k^{n-1}\,\mathbf{I} + \lambda_k^{n-2}\,\mathbf{B}_1 + \ldots + \mathbf{B}_{n-1}$$

Faddejev assumes for different eigenvalues, that each column of $\mathbf{Q}_k$ can be used as the respective eigenvector. This is not always true.

Horner recursion for the eigenvector matrix ($i = 1$ to $n - 1$):

$\mathbf{Q}_{k,0} = \mathbf{I}$

$\mathbf{Q}_{k,i} = \lambda_k \mathbf{Q}_{k,i-1} + \mathbf{B}_i$

$\mathbf{Q}_k = \mathbf{Q}_{k,n-1}$

3.1 Faddejev algorithm / Diagonal matrix with distinct eigenvalues

A diagonal matrix with diagonal values 1,2,3 delivers the eigenvalues 1,2,3. The roots of the characteristic equation are found in an unpredictable order. The matrix **Q** contains indeed for each eigenvalue the respective eigenvector: it is the only one non-zero column. In a program one cannot check for non-zero. Alternatively, the column with the largest norm is selected. Faddejev says, that any column can be used. This is correct, if the matrix does not consist of diagonal elements or diagonally arranged blocks.

Faddejev algorithm, arranged for manual calculations (n = 3) :

$$p_1 = \mathrm{tr}(\mathbf{A}_1)/1 \qquad p_2 = \mathrm{tr}(\mathbf{A}_2)/2 \qquad p_3 = \mathrm{tr}(\mathbf{A}_3)/3$$

$$\mathbf{B}_1 = \mathbf{A}_1 - p_1\mathbf{I} \qquad \mathbf{B}_2 = \mathbf{A}_2 - p_2\mathbf{I} \qquad \mathbf{B}_3 = \mathbf{A}_3 - p_3\mathbf{I}$$

$$\mathbf{A}_1 = \mathbf{A} \qquad \mathbf{A}_2 = \mathbf{A}\mathbf{B}_1 \qquad \mathbf{A}_3 = \mathbf{A}\mathbf{B}_2$$

$$p_1 = 6 \qquad p_2 = -11 \qquad p_3 = 6$$

$$\begin{bmatrix} -5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} -5 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -9 \end{bmatrix} \quad \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Determinant and inverse matrix:

$$\det \mathbf{A} = (-1)^{n-1} p_n = 6$$

$$\mathbf{A}^{-1} = \mathbf{B}_{n-1}/p_n = (1/6)\begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Characteristic equation:

$$f(\lambda) = (-1)^n [\lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \ldots - p_{n-1}\lambda^1 - p_n] = -[\lambda^3 - 6\lambda^2 + 11\lambda - 6] = 0$$

Eigenvalues, assumed to be found by this arbitrary order:

$$\lambda_1 = 3$$

$$\lambda_2 = 1$$

$$\lambda_3 = 2$$

Eigenvector matrix and normalized eigenvectors:

$$\mathbf{Q}_k = \lambda_k^{n-1}\mathbf{I} + \lambda_k^{n-2}\mathbf{B}_1 + \ldots + \mathbf{B}_{n-1}$$

$$\mathbf{Q}_k = \lambda_k^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \lambda_k\begin{bmatrix} -5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda_k^2 - 5\lambda_k + 6 & 0 & 0 \\ 0 & \lambda_k^2 - 4\lambda_k + 3 & 0 \\ 0 & 0 & \lambda_k^2 - 3\lambda_k + 2 \end{bmatrix}$$

$$\mathbf{Q}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \mathbf{n}_{10} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{Q}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{n}_{20} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{Q}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{n}_{30} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3.2 Faddejev algorithm / Matrix with complex eigenvalues

For complex eigenvalues the eigenvectors are complex as well. The norm for searching the largest eigenvector is based on the Euclidian norm of the absolute values of the complex numbers (squareroot of the sum of the squares of all real and imaginary numbers). For instance in $\mathbf{Q}_2$ one can use the first or the second column. Normalization is based on value 1 for the real part of one component.

Faddejev algorithm, arranged for manual calculations (n = 3) :

$$p_1 = \mathrm{tr}(\mathbf{A}_1)/1 \qquad p_2 = \mathrm{tr}(\mathbf{A}_2)/2 \qquad p_3 = \mathrm{tr}(\mathbf{A}_3)/3$$

$$\mathbf{B}_1 = \mathbf{A}_1 - p_1\mathbf{I} \qquad \mathbf{B}_2 = \mathbf{A}_2 - p_2\mathbf{I} \qquad \mathbf{B}_3 = \mathbf{A}_3 - p_3\mathbf{I}$$

$$\mathbf{A}_1 = \mathbf{A} \qquad \mathbf{A}_2 = \mathbf{A}\mathbf{B}_1 \qquad \mathbf{A}_3 = \mathbf{A}\mathbf{B}_2$$

$$p_1 = 3 \qquad p_2 = -4 \qquad p_3 = 2$$

$$\begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -3 & 1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Determinant and inverse matrix:

$$\det \mathbf{A} = (-1)^{n-1} p_n = 2$$

$$\mathbf{A}^{-1} = \mathbf{B}_{n-1}/p_n = (1/2)\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Characteristic equation:

$$f(\lambda) = (-1)^n [\lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \ldots - p_{n-1}\lambda^1 - p_n] = -[\lambda^3 - 3\lambda^2 + 4\lambda - 2] = 0$$

Eigenvalues, assumed to be found by this arbitrary order:

$$\lambda_1 = 1$$

$$\lambda_2 = 1 + j$$

$$\lambda_3 = 1 - j$$

Eigenvector matrix and normalized eigenvectors:

$$\mathbf{Q}_k = \lambda_k^{n-1}\mathbf{I} + \lambda_k^{n-2}\mathbf{B}_1 + \ldots + \mathbf{B}_{n-1}$$

$$\mathbf{Q}_k = \lambda_k^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \lambda_k\begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda_k^2 - 2\lambda_k + 1 & -\lambda_k + 1 & 0 \\ \lambda_k - 1 & \lambda_k^2 - 2\lambda_k + 1 & 0 \\ 0 & 0 & \lambda_k^2 - 2\lambda_k + 2 \end{bmatrix}$$

$$\mathbf{Q}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{n}_{10} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{Q}_2 = \begin{bmatrix} -1 & -j & 0 \\ j & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{n}_{20} = \begin{bmatrix} 1 \\ j \\ 0 \end{bmatrix}$$

$$\mathbf{Q}_3 = \begin{bmatrix} -1 & j & 0 \\ -j & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{n}_{30} = \begin{bmatrix} 1 \\ -j \\ 0 \end{bmatrix}$$

3.3 Faddejev algorithm / Matrix with equal eigenvalues

The example below shows in fact two cases. For all values of parameter k the eigenvalues are the same, and two of them are equal. For λ_1,λ_2, the eigenvectors cannot be found safely. For k=0 two linearly independent eigenvectors exist, but they cannot be calculated. For k≠0 both eigenvectors are equal. They can be calculated, but they do not form a linearly independent vector base.

Faddejev algorithm, arranged for manual calculations (n= 3) :

$$p_1 = \mathrm{tr}(\mathbf{A}_1)/1 \qquad p_2 = \mathrm{tr}(\mathbf{A}_2)/2 \qquad p_3 = \mathrm{tr}(\mathbf{A}_3)/3$$

$$\mathbf{B}_1 = \mathbf{A}_1 - p_1\mathbf{I} \qquad \mathbf{B}_2 = \mathbf{A}_2 - p_2\mathbf{I} \qquad \mathbf{B}_3 = \mathbf{A}_3 - p_3\mathbf{I}$$

$$\mathbf{A}_1 = \mathbf{A} \qquad \mathbf{A}_2 = \mathbf{A}\mathbf{B}_1 \qquad \mathbf{A}_3 = \mathbf{A}\mathbf{B}_2$$

$$p_1 = 4 \qquad p_2 = -5 \qquad p_3 = 2$$

$$\begin{bmatrix} -3 & k & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 2 & -2k & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} -3 & -2k & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Determinant and inverse matrix:

$$\det \mathbf{A} = (-1)^{n-1} p_n = 2$$

$$\mathbf{A}^{-1} = \mathbf{B}_{n-1}/p_n = (1/2)\begin{bmatrix} 2 & -2k & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

Characteristic equation:

$$f(\lambda) = (-1)^n [\lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \ldots - p_{n-1}\lambda^1 - p_n] = -[\lambda^3 - 4\lambda^2 + 5\lambda - 2] = 0$$

Eigenvalues, two are equal:

$$\lambda_1 = 1$$

$$\lambda_2 = 1$$

$$\lambda_3 = 2$$

Eigenvector matrix and normalized eigenvectors:

$$\mathbf{Q}_k = \lambda_k^{n-1}\mathbf{I} + \lambda_k^{n-2}\mathbf{B}_1 + \ldots + \mathbf{B}_{n-1}$$

$$\mathbf{Q}_k = \lambda_k^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \lambda_k\begin{bmatrix} -3 & k & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 2 & -2k & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_k^2 - 3\lambda_k + 2 & k(\lambda_k - 2) & 0 \\ 0 & \lambda_k^2 - 3\lambda_k + 2 & 0 \\ 0 & 0 & \lambda_k^2 - 2\lambda_k + 1 \end{bmatrix}$$

$$\mathbf{Q}_1 = \begin{bmatrix} 0 & -k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{n} = \begin{bmatrix} 1-\lambda_1 & k & 0 \\ 0 & 1-\lambda_1 & 0 \\ 0 & 0 & 2-\lambda_1 \end{bmatrix}\mathbf{n} = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{n} = \mathbf{0} \quad \mathbf{n}_{10} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{Q}_2 = \begin{bmatrix} 0 & -k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\mathbf{A} - \lambda_2\mathbf{I})\mathbf{n} = \begin{bmatrix} 1-\lambda_2 & k & 0 \\ 0 & 1-\lambda_2 & 0 \\ 0 & 0 & 2-\lambda_2 \end{bmatrix}\mathbf{n} = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{n} = \mathbf{0} \quad \mathbf{n}_{20} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{Q}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\mathbf{A} - \lambda_3\mathbf{I})\mathbf{n} = \begin{bmatrix} 1-\lambda_3 & k & 0 \\ 0 & 1-\lambda_3 & 0 \\ 0 & 0 & 2-\lambda_3 \end{bmatrix}\mathbf{n} = \begin{bmatrix} -1 & k & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{n} = \mathbf{0} \quad \mathbf{n}_{30} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

4.1 Rotation matrices / General

General rotation matrices are built by multiplying single axis matrices. The resulting compound matrices are under all circumstances orthonormal: the determinant is one, columns are orthonormal, and rows as well. The inverse matrix is easily found by transposing. An arbitrary 3D rotation by three *Euler* angles can be replaced by a single axis rotation. The rotation axis **n** is calculated either by the eigenvector for the eigenvalue λ_1=1 or the formula below. The axis is undefined for 0° and 180°.

Single axis rotation matrices:

Coordinate transformations

about x-axis, y-axis and z-axis:

$$\mathbf{x}_2 = \mathbf{X}_{21}\,\mathbf{x}_1$$

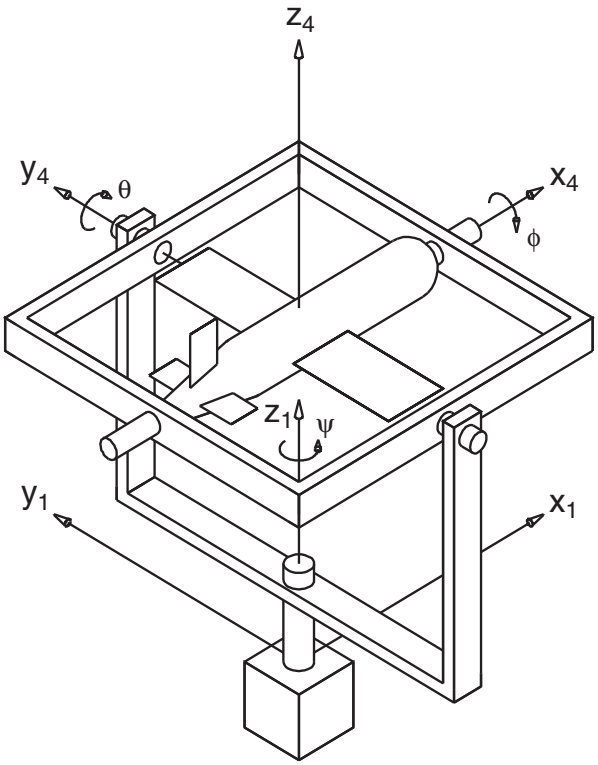
$$\mathbf{x}_2 = \mathbf{Y}_{21}\,\mathbf{x}_1$$

$$\mathbf{x}_2 = \mathbf{Z}_{21}\,\mathbf{x}_1$$

$$\mathbf{X}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$\mathbf{Y}_{21} = \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ \sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

$$\mathbf{Z}_{21} = \begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Coordinate transformation, Aircraft Euler angles:

$$\mathbf{x}_4 = \mathbf{A}_{41}\,\mathbf{x}_1$$

Compound rotation matrix, using ψ=yaw, θ=pitch and ϕ=roll :

$$\mathbf{A}_{41} = \mathbf{X}_{43}(\phi)\,\mathbf{Y}_{32}(\theta)\,\mathbf{Z}_{21}(\psi)$$

$$\mathbf{A}_{41} = \begin{bmatrix} \cos(\theta)\cos(\psi) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ -\cos(\phi)\sin(\psi)+\sin(\phi)\sin(\theta)\cos(\psi) & \cos(\phi)\cos(\psi)+\sin(\phi)\sin(\theta)\sin(\psi) & \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\psi)+\cos(\phi)\sin(\theta)\cos(\psi) & -\sin(\phi)\cos(\psi)+\cos(\phi)\sin(\theta)\sin(\psi) & \cos(\phi)\cos(\theta) \end{bmatrix}$$

The *Euler theorem* says:

Each rotation in 3D by a matrix **R** can be replaced by an equivalent single axis rotation with an angle η about an axis **n**. The transformations were introduced above as coordinate transformations.

For a rotation we have to use the inverse (transposed) matrix $\mathbf{R} = \mathbf{A}_{41}^{\mathrm{T}}$.

Angle and axis:

$$\eta = \arccos[0.5(r_{11}+r_{22}+r_{33}-1)]$$

$$\mathbf{n} = \frac{0.5}{\sin(\eta)}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

A rotation matrix has always one eigenvalue $\lambda_1 = 1$. **n** is the normalized eigenvector for λ_1.

For $\sin(\eta) = 0$ the axis **n** is undefined. This happens for $\eta = 0$ (no rotation) and $\eta = 180°$.

4.2 Rotation matrices / Eigenvalues and eigenvectors

A single rotation about the z-axis shows already some fundamental features of rotation matrices. For better understandability the standard eigenvalue equations are shown as well.

Faddejev algorithm for a rotation matrix:

Rotation about the z-axis, using $c = \cos(\psi)$ and $s = \sin(\psi)$

$$p_1 = 1 + 2c \qquad p_2 = -(1 + 2c) \qquad p_3 = 1$$

$$\begin{bmatrix} -(1+c) & -s & 0 \\ s & -(1+c) & 0 \\ 0 & 0 & -2c \end{bmatrix} \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -(1+c) & s & 0 \\ -s & -(1+c) & 0 \\ 0 & 0 & -2c \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinant and inverse matrix:

$$\det \mathbf{A} = (-1)^{n-1} p_n = 1$$

$$\mathbf{A}^{-1} = \mathbf{B}_{n-1} / p_n = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{A}^{T}$$

Characteristic equation:

$$f(\lambda) = (-1)^n [\lambda^n - p_1 \lambda^{n-1} - p_2 \lambda^{n-2} - \ldots - p_{n-1} \lambda^1 - p_n] = -[\lambda^3 - (1+2c)\lambda^2 + (1+2c)\lambda - 1] = 0$$

Eigenvalues, two are conjugate complex:

$$\lambda_1 = 1$$

$$\lambda_2 = c + j\,s$$

$$\lambda_3 = c - j\,s$$

Eigenvector matrix and normalized eigenvectors:

$$\mathbf{Q}_k = \lambda_k^{n-1} \mathbf{I} + \lambda_k^{n-2} \mathbf{B}_1 + \ldots + \mathbf{B}_{n-1} = \begin{bmatrix} \lambda_k^2 - (1+c)\lambda_k + c & s(1-\lambda_k) & 0 \\ -s((1-\lambda_k) & \lambda_k^2 - (1+c)\lambda_k + c & 0 \\ 0 & 0 & \lambda_k^2 - 2c\lambda_k + 1 \end{bmatrix}$$

$$\mathbf{Q}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2(1-c) \end{bmatrix} \qquad \mathbf{n}_{10} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad c \neq 1$$

$$\mathbf{Q}_2 = \begin{bmatrix} -s^2 - j\,s(1-c) & s(1-c) - j\,s^2 & 0 \\ -s(1-c) + j\,s^2 & -s^2 - j\,s(1-c) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -s - j\,(1-c) & (1-c) - j\,s & 0 \\ -(1-c) + j\,s & -s - j\,(1-c) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{n}_{20} = \begin{bmatrix} 1 \\ -j \\ 0 \end{bmatrix} \quad s \neq 0$$

$$\mathbf{Q}_3 = \begin{bmatrix} -s^2 + j\,s(1-c) & s(1-c) + j\,s^2 & 0 \\ -s(1-c) - j\,s^2 & -s^2 + j\,s(1-c) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -s + j\,(1-c) & (1-c) + j\,s & 0 \\ -(1-c) - j\,s & -s + j\,(1-c) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{n}_{30} = \begin{bmatrix} 1 \\ j \\ 0 \end{bmatrix} \quad s \neq 0$$

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{n} = \begin{bmatrix} c-\lambda_1 & -s & 0 \\ s & c-\lambda_1 & 0 \\ 0 & 0 & 1-\lambda_1 \end{bmatrix} \mathbf{n} = \begin{bmatrix} c-1 & -s & 0 \\ s & c-1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{n} = \mathbf{0} \qquad \mathbf{n}_{10} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{n} = \begin{bmatrix} c-\lambda_2 & -s & 0 \\ s & c-\lambda_2 & 0 \\ 0 & 0 & 1-\lambda_2 \end{bmatrix} \mathbf{n} = \begin{bmatrix} -j\,s & -s & 0 \\ s & -j\,s & 0 \\ 0 & 0 & (1-c) - j\,s \end{bmatrix} \mathbf{n} = \mathbf{0} \qquad \mathbf{n}_{20} = \begin{bmatrix} 1 \\ -j \\ 0 \end{bmatrix} \quad s \neq 0$$

$$(\mathbf{A} - \lambda_3 \mathbf{I})\mathbf{n} = \begin{bmatrix} c-\lambda_3 & -s & 0 \\ s & c-\lambda_3 & 0 \\ 0 & 0 & 1-\lambda_3 \end{bmatrix} \mathbf{n} = \begin{bmatrix} j\,s & -s & 0 \\ s & j\,s & 0 \\ 0 & 0 & (1-c) + j\,s \end{bmatrix} \mathbf{n} = \mathbf{0} \qquad \mathbf{n}_{30} = \begin{bmatrix} 1 \\ j \\ 0 \end{bmatrix} \quad s \neq 0$$

5.1 Newton-Bairstow algorithm

The *Newton-Bairstow* algorithm [2] finds all roots of a polynomial with real coefficients.

Polynomial with degree n with real coefficients:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ...a_1 x^1 + a_0$$

Find all roots $x_1 ... x_n$ for

$$f(x) = 0$$

Find a quadratic divisor:

$$p(x) = x^2 - r\,x - s$$

Solve quadratic equation, using the *Hoffmann* algorithm [8].
Results are two, one or no roots.
This algorithm is protected against division overflow.
Two roots are either real or conjugate complex.

Find the reduced polynomial with degree $m = n - 2$:

$$g(x) = b_m x^m + ... + b_1 x_1 + b_0$$

Repeat steps until $m \leq 2$.

The iteration is executed by a double *Horner* algorithm [7].

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